GFD Fundamentals

$\lambda \rightarrow$ longitude

$\theta \rightarrow$ latitude

$f = 2\Omega \sin\theta$ Coriolis parameter

$\Omega = 7.3x10^{-5} s^{-1}$ Earth rotation

$R_0 = \dfrac{U}{fL}$ Rossby # [< 1 for wide range of motions]

Equations in non-rotating frame:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0 & \text{continuity} \\ \rho \dfrac{D\mathbf{u}}{Dt} = -\nabla p + \rho \nabla \phi + F(\mathbf{u}) & \text{momentum} \\ \text{energy} & \\ & \end{cases}$$

$$F(\mathbf{u}) = \mu \nabla^2 \mathbf{u} + \frac{\mu}{3} \nabla(\nabla \cdot \mathbf{u}) \qquad \text{Frictional force}$$

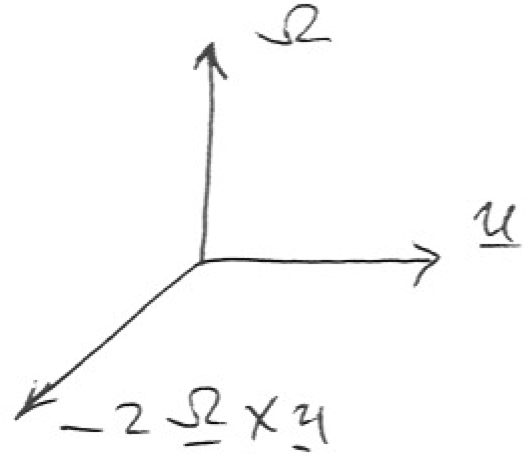
Equations in rotating frame:

$$\rho\left[\frac{D\mathbf{u}}{Dt}+2\Omega\times\mathbf{u}\right]=-\nabla p+\rho\nabla\Phi+F(\mathbf{u})$$

$$\Phi=\phi+\frac{1}{2}\left|\Omega\times\mathbf{r}\right|^2$$

- Coriolis force is $\perp$ to $\mathbf{U}$ and hence <u>does no work.</u>

- Total derivative of all scalars (density, energy) are unchanged in rotating frame.



(Approximate) Equations of motion:

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \left(f + u\frac{\tan\theta}{a} \right) v + F_\lambda$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \left(f + \frac{u\tan\theta}{a} \right) u + F_\theta$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_z$$

Here x, y, z are curvilinear coordinates along east, north and local vertical directions. F_λ, F_θ and F_z are frictional forces towards east, etc.; a is Earth radius.

f-plane model

Coriolis parameter $f = 2\Omega\sin\theta$ varies with latitude. Variation is important only for very long time scales or very long length scales (several weeks or $>10^3$ km) and otherwise $f_0 = 2\Omega\sin\theta_0$ can be taken in model.

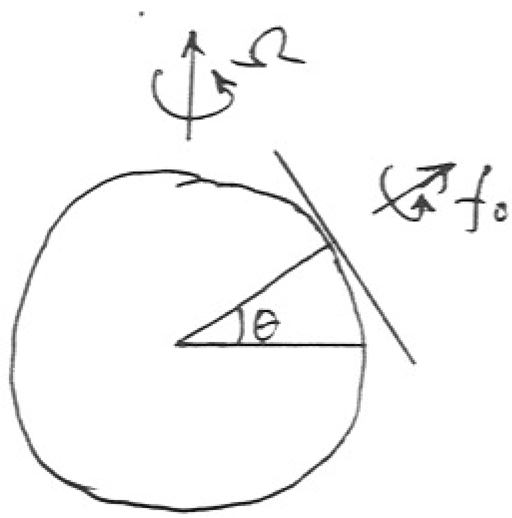
β-plane model

Alternatively one can take a linear approximation to f as

$$f = f_0 + \beta y$$

$$\beta = \frac{2\Omega\cos\theta_0}{a}$$

and such models are referred to as β-plane models.



Taylor-Proudman Theorem

Homogeneous, frictionless fluid,

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$g = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Eliminating p gives $f\left(\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right) = 0$, using continuity this implies

$$\frac{\partial w}{\partial z} = 0$$

Also differentiating v and u equations by z gives

$$\frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} = 0$$

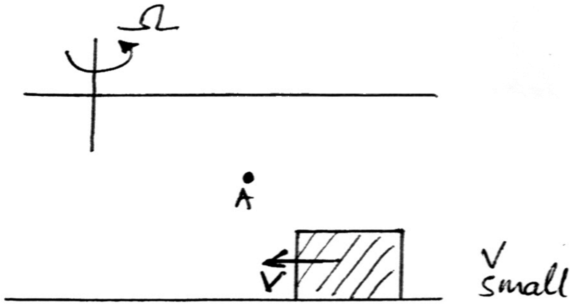
or

$$\frac{\partial \mathbf{u}}{\partial z} = 0 \qquad \text{Taylor-Proudman Theorem}$$

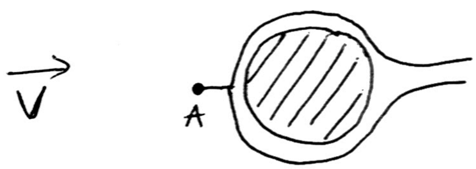
Consequence of Taylor-Proudman Theorem

Horizontal velocity field has no vertical shear and motion is two dimensional (in columns).

For dye released at A above the cylinder, the dye goes around the imaginary cylinder above the actual cylinder.



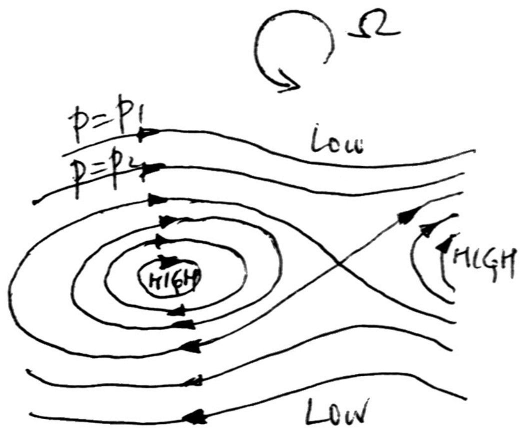
G.I. Taylor expt



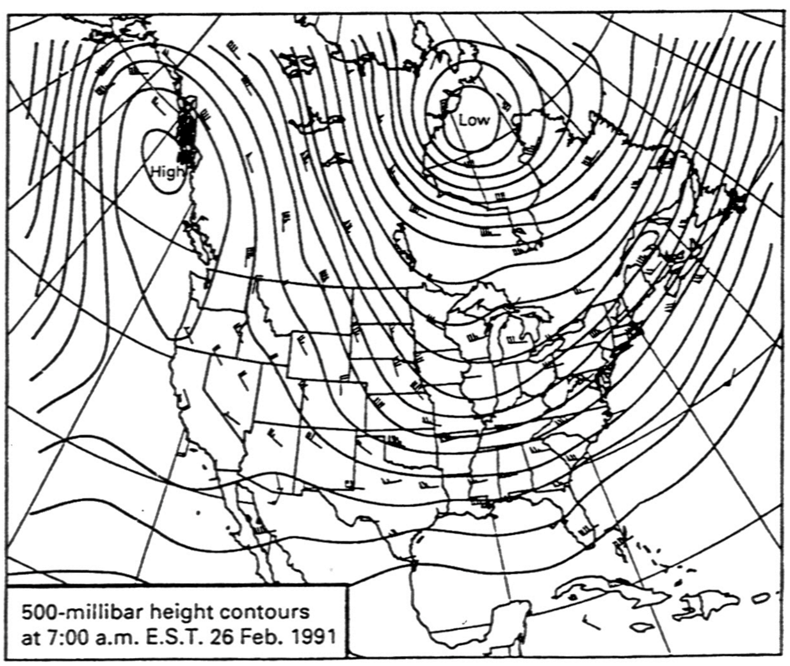
– Geostrophic flow

Also note that $u = -\dfrac{1}{\rho f}\dfrac{\partial p}{\partial y}$, $v = \dfrac{1}{\rho f}\dfrac{\partial p}{\partial x}$ and $\mathbf{u}_H$ is $\perp$ to $\left(\dfrac{\partial p}{\partial x}, \dfrac{\partial p}{\partial y}\right)$, which is the pressure gradient. Thus motion is not *down-gradient* but *across-gradient*. Fluid particles do not move from high pressure to low pressure as in an inviscid non-rotating flow but move along isobars, which are (approximate) streamlines.

In the Northern Hemisphere, with a counterclockwise mean rotation, implies f is positive and currents flow with high pressure to their right.



Northern Hemisphere



500-millibar height contours at 7:00 a.m. E.S.T. 26 Feb. 1991

Vorticity dynamics

$$\Gamma = \iint_A \omega \cdot \mathbf{n}\, dA = \oint_c \mathbf{u} \cdot d\mathbf{r}$$

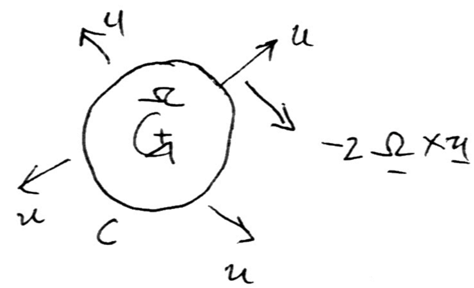
Circulation measures strength of vortex-tube

$$\frac{d\Gamma}{dt} = \oint_c (2\Omega \times \mathbf{u}) \cdot d\mathbf{r} - \oint_c \frac{\nabla p}{\rho} \cdot d\mathbf{r} + \oint_c \frac{F}{\rho} \cdot d\mathbf{r}$$

Circulation of relative vorticity (absolute vorticity $\omega_a = \omega + 2\Omega$) can increase due to these three effects.

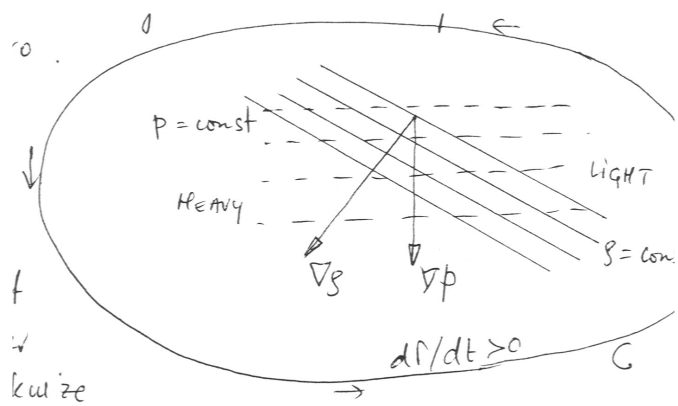
1. Coriolis force

$$\frac{d\Gamma}{dt} < 0$$



Also in the presence of planetary vorticity, relative motions induct this and large scale motions are usually not free of relative vorticity.

2. Pressure term



$$-\oint_c \frac{\nabla \mathrm{p}}{\rho} \cdot d\mathbf{r} = \iint_A \nabla \times \left(\frac{\nabla \mathrm{p}}{\rho} \right) \cdot \mathbf{n}\, dA$$

$$= \iint_A \frac{\nabla \rho x \nabla p}{\rho^2} \cdot \mathbf{n}\, dA$$

If surfaces of constant pressure do <u>not</u> coincide $\Rightarrow$ <u>baroclinic flow</u>.

If the flow is baroclinic, relative circulation will change if average normal component on A is non-zero.

Light and heavy fluids experience the same force, and so the light fluid will rise faster creating counterclockwise circulation.

If the density field is convected with the fluid, the resulting circulation will align the density and pressure surfaces.

3. Frictional effects are diffusive (heat analogy).

<u>Non-geostrophic flow</u>

If fluid is <u>not</u> rotating rapidly enough, and other acceleration terms cannot be neglected compared to coriolis, but still considered homogeneous and inviscid

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}-fu=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial z}=0 \quad \text{considering departures from hydrostatic equilibrium}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

- If u, v are independent of z initially, they will remain so.
- This is <u>barotropic flow</u>:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-fu=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

This is superficially similar to geostrophic flow in that there is no vertical structure, but here the flow is not required to be aligned with the isobars and can possess vertical velocity.

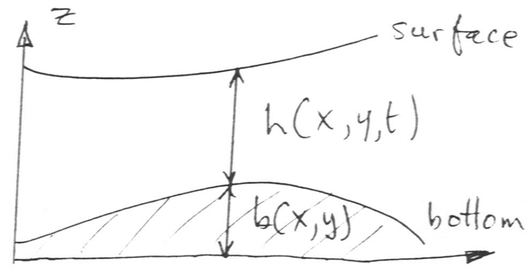
$$\frac{\partial w}{\partial z}=-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\neq 0$$

A vertical velocity varying linearly with depth can exist enabling the flow to support flow across isobaths (surfaces of constant depth).

Integrating

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\int_b^{b+h} dz+[w]_b^{b+h}=0$$

with



$$=\frac{\partial}{\partial t}(b+h)+u\frac{\partial}{\partial x}(b+h)+v\frac{\partial}{\partial y}(b+h)$$

$$w_{z=b}=u\frac{\partial b}{\partial x}+v\frac{\partial b}{\partial y}$$

so that we have

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial n}(hu)+\frac{\partial}{\partial y}(hv)=0$$

and w has been eliminated.

Since the fluid is homogeneous, p is independent of z. For absence of pressure variation above fluid surface (e.g. uniform atmospheric pressure over the ocean), dynamic pressure is

$$p=\rho_0 g(h+b)$$

For flat bottom we have

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv=-g\frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial n}+v\frac{\partial v}{\partial y}-fu=-g\frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}(hu)+\frac{\partial}{\partial y}(hv)=0$$

Which are they well known <u>Shallow water equations.</u>

Conservation of potential vorticity

If we retain the acceleration terms by $-\partial_y B_1 + \partial_x B_2$ to obtain

$$\frac{D}{Dt}\left(f+\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(f+\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=0$$

$$f+\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=f+\zeta$$

Since the horizontal flow has no depth dependence, there is no vertical shear and hence no horizontal vorticity

or $$\frac{D}{Dt}\left(f+\zeta\right)+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)(f+\zeta)=0$$

Also can write the continuity equation as

$$\frac{D}{Dt}h+\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)h=0$$

And hence

$$\frac{D}{Dt}\left(\frac{f+\zeta}{h}\right)=0$$

$q=\dfrac{f+\zeta}{h}$ is potential vorticity and is conserved.

For rapidly rotating flows $\left(R_0=\dfrac{U}{\Omega L}\ll 1\right)$ have $\zeta \ll f$ and

$$q=\frac{f}{h}$$

and if $f \simeq const$ (geophysical flow of modest extent)

each fluid column must conserve its height h. If the top is horizontal, then fluid parcels must follow isobaths.

Primitive equation AGCM

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \cdot \nabla \mathbf{u} - \omega \frac{\partial \mathbf{u}}{\partial p} + f\,\mathbf{k} \times \mathbf{u} - \nabla \Phi + F_M$$

$$\frac{\partial T}{\partial t} = -u \cdot \nabla T - \omega \frac{kT}{p} - \frac{\partial T}{\partial p} + \frac{\tilde{Q}_{rad}}{C_p} + \frac{\tilde{Q}_{con}}{C_p} + F_H$$

$$\frac{\partial q}{\partial t} = -\mathbf{u} \cdot \nabla q - \omega \frac{\partial q}{\partial p} + E - C + F_q$$

$$\frac{\partial \omega}{\partial p} = -\nabla \cdot u$$

$$\frac{\partial \Phi}{\partial p} = -\frac{RT}{p}$$

$F_M \rightarrow$ friction term for momentum

$F_H, F_q \rightarrow$ friction term for heat and moisture

$q \rightarrow$ specific humidity

$E, C \rightarrow$ rates of evaporation and condensation (clouds)

$\Phi \rightarrow$ geopotential

$\tilde{Q}_{rad}, \tilde{Q}_{con}, E, C$ “model physics”

- zero, 1D, 2D, full models.